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Wh $[m]$. *A whist-tournament arrangement in m players Wh* $[m]$ $(m \equiv 0, \text{mod. } 4)$ *is an arrangement of* $m-1$ *evenings of play in such a way that every player meets in play every other player once as partner and twice as opponent.* The understanding is that there is involved *no question of order* of the (1) evenings, (2) tables, and (3) players at a table, except as indicated by the partnership pairs. In accordance with (3) we use the notation $|a_i a_j a_k a_l|$ for a say *table* of players; this table is unchanged if we make on the four letters certain eight substitutions, those of the group G_8^4 which preserve the pairings $a_i a_j$, $a_k a_l$.

Whist-tournament arrangements *Wh* $[m]$ have been constructed[*] by Messrs. Mitchell, Safford, Howell, and Whitfeld for the cases

$$m = 4, 8, 12, 16, 20, 24, 28, 32, 36, \text{ and } 40.$$

No general theorems or methods have heretofore been published.

2. *Cyclic Wh* $[m]$. A *Wh* $[m]$ is called *cyclic* if the $m-1$ evenings are obtained from the first by repetitions of a certain say *cyclic substitution* σ which leaves one letter fixed and permutes the $m-1$ remaining in one cycle σ. (No confusion will arise from using the same notation for the cyclic substitution and its cycle.) For the letters of a cyclic *Wh* $[m]$ the *index* notation is the clearest. The m letters or indices are $*, 0, 1, 2, \ldots . \overline{m-2}$; the substitution $\sigma \equiv (i' = i + 1)$ leaves $*$ fixed and replaces i by $i+1$ $(i = 0, 1 \ldots . \overline{m-2})$, the indices being taken modulo $m-1$; in this index notation the cyclic *Wh* $[m]$ is given by its first evening.

The arrangements heretofore constructed are[†] *with one exception all cyclic,* and they were for the most part so constructed. We have, e. g.,

Wh $[4]$ $|*\ 0\ 1\ 2|$. (Only class.)
Wh $[8]$ $|*\ 0\ 4\ 5|$ $|1\ 3\ 2\ 6|$. (Safford: B, p. 119.) (Only class.)
Wh $[12]_1$ $|*\ 0\ 4\ 5|$ $|1\ 10\ 2\ 8|$ $|3\ 7\ 6\ 9|$. (Safford: B, p. 119.)
Wh $[12]_2$ $|*\ 0\ 2\ 3|$ $|1\ 8\ 5\ 7|$ $|4\ 10\ 6\ 9|$. (New class.)
Wh $[16]$ $|*\ 0\ 10\ 5|$ $|1\ 2\ 4\ 8|$ $|3\ 11\ 14\ 12|$ $|6\ 9\ 13\ 7|$. (Safford: B, p. 119.)

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[*] For the published literature of the subject see :

A. "Duplicate Whist," by John T. Mitchell (McClurg, Chicago, 1891), pp. 17, 18, 21.

B. The periodical "Whist," vol. 1, pp. 101, 119, 123, 149, 1891-2 (Cassius M. Paine, editor, Milwaukee, Wis.).

C. "Duplicate Whist and Whist Strategy," by R. F. Foster (Brentano, 1894), pp. 64, 68, 71.

I shall cite these sources by the letters A, B, C.

Mr. Mitchell called my attention to the problem. In the second edition of his "Duplicate Whist," a chapter on this subject by Mr. Whitfeld will be incorporated.

[†] As Mr. Mitchell has informed me.

Let us have a (first) evening given in this index notation. From it we write down in all $m-1$ evenings. Have we now a $Wh\,[m]$? The $*$ assuredly meets every other player once as partner, twice as opponent. Between every two indices $i_1, i_2\ (i_1 \neq i_2)$ there are on the face of the cycle σ two *intervals* $\pm(i_1-i_2)$, modulo $m-1$; these are distinct since $m-1$ is odd. There are in all $m-2$ intervals. We compute from the given first evening two interval-systems $(IS)_p$ and $(IS)_o$, containing the intervals between $(IS)_p$ every pair of partners, $(IS)_o$ every pair of opponents. [Since an interval is invariant under the cyclic substitution, these interval-systems have no particular relation to the first evening.] The interval-system $(IS)_p$ contains $m-2$ intervals; if every possible interval is present ($\therefore$ without repetition), in the $m-1$ evenings' play every player meets every other player once as partner. The interval-system $(IS)_o$ contains $2(m-2)$ intervals; if every possible interval is present twice ($\therefore$ exactly twice), in the $m-1$ evenings' play every player meets every other player twice as opponent. The given first evening leads then to a (cyclic) $Wh\,[m]$ if and only if the two interval-systems satisfy the two conditions just explained.

3. *The substitution-group G^m of a cyclic $Wh\,[m]$ ($m-1$ prime).* The substitutions on the m letters leaving any $Wh\,[m]$ invariant form a group G^m; those substitutions of G^m leaving any index i invariant form a sub-group $G^{m;\,i}$. For the *cyclic* $Wh\,[m]$ the G^m is transitive either on all m letters or on $m-1$ letters. In the latter case every substitution leaves $*$ fixed; G^m and $G^{m;\,*}$ are identical. In the former case the m sub-groups $G^{m;\,i}$ $(i=*, 0, 1 \ldots . \overline{m-2})$ are conjugate under G^m. We study then primarily the sub-group $G^{m;\,*}$ We assume that $m-1$ is prime.

The $G^{m;\,*}$ contains the fundamental cyclic substitution σ of the cyclic $Wh\,[m]$, and hence the cyclic sub-group $G_{m-1}\{\sigma\}$ of order $m-1$. Now the $Wh\,[m]$ *of itself* defines a say *natural* cycle of the $m-1$ indices ($*$ apart). In this way: The $Wh\,[m]$ contains say the table $|*\ g\ 0\ h|$, and hence the tables $|*\ f+g\ f\ f+h|$ $(f=0, 1, \ldots . \overline{m-2})$. The cycle in question is $0, h, 2h, \ldots$ $kh, \ldots . (m-2)h$; this cycle contains the $m-1$ indices in a definite (perhaps reversible) cyclic order; any two consecutive letters $kh\ (k+1)h$ of the cycle are in the $Wh\,[m]$ paired as partners against $*$, in the table $|*\ kh+g\ kh\ (k+1)h|$. Any substitution of $G^{m;\,*}$ throws this cycle into itself, perhaps reversing the cyclic order. The $G^{m;\,*}$ is obtained from the $G_{m-1}\{\sigma\}$ by extending it by the substitutions τ of the group $G^{m;\,*0}$, leaving $*$ and 0 each fixed. Obviously any such τ is of the form $i' = i$ or $i' = -i$. Hence the $G^{m;\,*}$ is either the cyclic group $G_{m-1}\{\sigma\}$, $i' = i + f \; (f = 0, 1 \ldots \overline{m-2})$, or the dihedron group $G_{2(m-1)}$, $i' = \pm\, i + f \; (f = 0, 1 \ldots . \overline{m-2})$.

The $G^{m;\,*}$ is indeed the $G^{m;\,*}_{2(m-1)}$ if and only if the substitution $i' = -i$ throws the (initial) evening with the table $|*\ 0\ -g\ h-g|$ into itself; for this, a necessary condition is that $h \equiv 2g \pmod{m-1}$.

The natural cycle given above corresponds to the cyclic substitution σ^h, which is therefore a *natural* fundamental cyclic substitution of the cyclic *Wh* $[m]$; σ^{-h} plays the same rôle. This remark furnishes a convenient *normal form* for the cyclic *Wh* $[m]$ ($m-1$ prime): *the opponents of $*$ are always consecutive indices*. (The *Wh* $[m]$, $m = 4, 8, 12$, given above are exhibited in this normal form.)

Now if the G^m is transitive on the m letters there must be a corresponding natural cycle with respect to every index. For any particular *Wh* $[m]$ it is easy to see whether the opponents of say 0 do determine such a natural cycle, and if so whether the corresponding cyclic substitution does throw the *Wh* $[m]$ into itself. If such a substitution exists, it extends the $G^{m;\,*}_{m-1 \text{ or } 2(m-1)}$ to the $G^m = G^m_{m(m-1) \text{ or } 2m(m-1)}$; if not, the G^m is identical with the $G^{m;\,*}_{m-1 \text{ or } 2(m-1)}$.

The $G^m_{m(m-1)}$ is exactly doubly transitive on the m letters, and hence (II **11** a) can occur only if $m = 2^{n+1}$, $m-1$ prime.

As to our examples:

Wh $[4]$ $\quad G^{4;\,*}_{6=2.3}$. $\quad G^4_{24=4.6}$. $\quad$ Extender $(0)(*\ 1\ 2)$.
Wh $[8]$ $\quad G^{8;\,*}_{7}$. $\quad G^8_{56=8.7}$. $\quad$ Extender $(0)(*\ 2\ 4\ 5\ 1\ 6\ 3)$.
Wh $[12]_1$ $\quad G^{12;\,*}_{11} = G^{12}_{11}$.
Wh $[12]_2$ $\quad G^{12;\,*}_{11} = G^{12}_{11}$.

There is no cyclic *Wh* $[m]$ with $G^{m;\,*} = G^{m;\,*}_{2(m-1)}$ for $m = 8, 12$ or 20.

4. *Concerning the substitution-group G^m of a cyclic Wh $[m]$ ($m-1$ composite).* I call attention briefly to certain *natural cycles, chains, rulings* of the $m-1$ letters ($*$ apart) which play a fundamental rôle in the determination of the $G^{m;\,*}$ and of the G^m. These depend upon the tables $|*\ f+g\ f\ f+h|$ $(f = 0, 1 \ldots . \overline{m-2})$ containing $*$. Denote by d', d'', d''' the greatest common divisors $d' = [h, m-1]$, $d'' = [g, h, m-1]$, $d''' = [2g-h, m-1]$.

As before a *cycle* has the characteristic property that every two consecutive letters are paired as partners against $*$. f lies in the cycle $f+kh\,(k=0,\,1,\ldots)$. The $m-1$ letters break into d' cycles of $(m-1)/d'$ letters each. The $(m-1)/d'$ letters of a cycle have a definite (perhaps reversible) cyclic order.

But $f\,f+h$ at the table $|*\ f+g\ f\ f+h|$ are paired also against $f+g$. In this way the cycle containing f *rests upon* the cycle containing $f+g$, and it in turn upon another cycle, and so on. In this way the d' cycles break into d'' *chains* of d'/d'' cycles and $(m-1)/d''$ letters each. The d'/d'' cycles of a chain have a definite and irreversible cyclic order.

On a chain every letter f belongs to a certain *ruling*. In the tables containing $*$, f opposes $*$ at the two tables $|*\ f+g\ f\ f+h|$, $|*\ f+g-h\ f-h\ f|$; f in its cycle rests upon the sequence $f+g-h\ f+g$ in the succeeding cycle, and in view of the table $|*\ f+2g-h\ \ f+g-h\ \ f+g|$ this sequence $f+g-h\ \ f+g$ rests upon $f+2g-h$ in the third cycle. f belongs to the *ruling* $f+k(2g-h)\ (k=0,\,1\,\ldots.)$.

In this way the $m-1$ letters break into d''' rulings of $(m-1)/d'''$ letters each. The $(m-1)/d'''$ letters of a ruling have a definite and irreversible cyclic order. Every chain contains d'''/d'' rulings. A ruling cuts out $(m-1)\,d''/d'd'''$ letters from every cycle of the chain; this number is 1 if and only if $(2g-h)\,d'/d''\equiv 0 \pmod{m-1}$; except in this case, the number is $\geqq 3$, *and the irreversible cyclic order on the ruling establishes*[*] *an irreversible cyclic order on every cycle of the chain.* Even if $(2g-h)\,d'/d''\equiv 0 \pmod{m-1}$, the cyclic orders of the d'/d'' cycles of the chain depend upon the cyclic order of any one of the cycles, so that we may speak of the cyclic order of the chain itself.

5. *$Wh\,[m]$ associated*[†] *with $SGS\,[4,\,2,\,m]$* $(\therefore\ m=12\nu+4,\ \nu\geqq 0$ (II 8 c)). A $SGS\,[4,\,2,\,m]$ is (II 6) an arrangement of the m letters in $\frac{1}{12}m\,(m-1)$ 4-ads so that every two letters meet once in a 4-ad, and then a separation of these 4-ads into $\frac{1}{3}(m-1)$ sets s_1 of $\frac{1}{4}m$ 4-ads each, every letter appearing once in the 4-ads of every set s_1. From every such $SGS\,[4,\,2,\,m]$ we can construct[†] in all $6^{\frac{1}{12}(m-4)(m-1)}$ *associated* $Wh\,[m]$. Every 4-ad $[a_i\,a_j\,a_k\,a_l]$ yields three evenings'

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[*] For $m-1$ prime, $\therefore\ d'=d''=d'''=1$, unless $2g-h\equiv 0 \pmod{m-1}$, the $d'=1$ cycle has an irreversible cyclic order (as we saw in §3).

[†] The only non-cyclic $Wh\,[m]$ heretofore constructed is Mr. Mitchell's $Wh\,[16]$ (A., p. 21) which is, as I say, associated with the (unique) $SGS\,[4,\,2,\,16]$. Of the 6^{15} $Wh\,[16]$ thus virtually given certain are surely cyclic (e. g. Mr. Safford's cyclic $Wh\,[16]$, B, p. 119), but it is quite unlikely that all are.

play for these four players, viz. at the tables $|a_i a_j a_k a_l|$, $|a_i a_k a_l a_j|$, $|a_i a_l a_j a_k|$. Every set s_1 yields thus (in $6^{\frac{1}{4}(m-4)}$ ways) three evenings' play for the m players. Any $m-1$ evenings' play so obtained from the $\frac{1}{3}(m-1)$ sets s_1 of the $SGS[4, 2, m]$ constitutes obviously a $Wh[m]$.

Thus we can construct a $Wh[4^n]$ (*n any integer*) associated with the $SGS[4, 2, 4^n]$ derived from the Galois-field $GF[4^n]$ (II 10 g), and a $Wh[4, 2, m = 3p + 1]$ (*p any prime of the form* $p = 4\nu + 1$) associated with the $SGS[4, 2, m = 3p + 1]$ to be constructed in §6 (II 10 h).

Again, we can compound two given $Wh[m_1]$, $Wh[m_2]$ associated with $SGS[4, 2, m_1]$, $SGS[4, 2, m_2]$ respectively into a $Wh[m_1 m_2]$ associated with the compound $SGS[4, 2, m_1 m_2]$ (II 16 i).

6. *Cyclic* $SGS[4, 2, m = 3p + 1]$ *where* p *is any prime of the form* $p = 3\nu + 1$. Before constructing this arrangement it is convenient to consider briefly more generally—

Cyclic $SGS[k, 2, m]$; $m = k(\overline{k-1}\,\nu + 1) = (k-1)p + 1$, *p any integer of the form* $p = k\nu + 1$ (II 8 c). We call a $SGS[k, 2, m]$ cyclic if for a certain index notation (§2) of the m letters the substitution $\sigma \equiv (i' = i + 1)$ permutes the $p = (m-1)/(k-1)$ sets $s_1 \equiv SGS[k, 1, m]$ cyclically. Then $\sigma^p \equiv (i' = i + p)$ throws every $SGS[k, 1, m]$ and also the $*$ into itself, and thus every k-ad containing $*$ into itself. Thus $*$ appears in the k-ad $[*\ vp\ (v = 0, 1, \ldots \overline{k-2})]$ in say the first $SGS[k, 1, m]$. This remark determines the p k-ads which contain $*$. The index u appears in the k-ad $[*\ u + vp\ (v = 0, 1, \ldots . \overline{k-2})]$ with *every* index u' whose forward interval from u on the face of the cycle σ, viz. $u' - u$ (mod. $m - 1 = (k-1)p$) is divisible by p. Hence these intervals vp occur *only* in the k-ads containing $*$ The original $SGS[k, 2, m]$ contains besides the k-ad $[*\ vp]$ $(k-1)\nu$ k-ads; these $(k-1)\nu$ k-ads break into ν sets of $k-1$ k-ads each; if one k-ad is $[i_1' \ldots . i_f' \ldots . i_k']$, or say $[[i_f']]$, the $k-1$ k-ads of its set are $[[i_f]]$ where

$$i_f \equiv i_f' + vp \text{ (mod. } kp) \; (f = 1 \ldots . k). \qquad (v = 0 \ldots . \overline{k-2})$$

We denote the $(k-1)k\nu = m - k$ indices of these $(k-1)\nu$ k-ads by $i_{fgv}\begin{pmatrix} f = 1 \ldots k \\ g = 1 \ldots . \nu \\ v = 0 \ldots . k-2 \end{pmatrix}$, so that the k indices i_{fgv} $(f = 1 \ldots . k)$ constitute a k-ad $[[i_f]]_{gv}$, and the $k-1$ k-ads $[[i_f]]_{gv}$ $(v = 0 \ldots . k-2)$ belong to such a set, and the ν sets are obtained by varying $g\,(g=1\,\ldots\,.\,\nu)$. We set in particular $i_{fg}=i_{fg0}\left(\begin{smallmatrix}f=1\,\ldots\,.\,k\\ g=1\,\ldots\,.\,\nu\end{smallmatrix}\right)$, $i_{fgv}\equiv i_{fg}+vp\ (\text{mod. } kp)\ (v=0\,\ldots\,.\,k-2)$, and $[[i_f]]_{g0}=[[i_f]]_g$.

Suppose, in an effort to construct a cyclic $SGS\,[k,\,2,\,m]$, we succeed in distributing to our m indices $*\ 0\,\ldots\,.\,\overline{m-1}$ the notation $*\ vp\ i_{fgv}$ subject to the preceding conditions. Then the $(k-1)\,\nu+1$ k-ads $[*\ vp]\ [[i_f]]_{gv}$ $\left(\begin{smallmatrix}g=1\,\ldots\,.\,\nu\\ v=0\,\ldots\,.\,k-2\end{smallmatrix}\right)$ constitute a $SGS\,[k,\,1,\,m]$ in the desired way invariant under σ^p.

From this we obtain in all p $SGS\,[k,\,1,\,m]$ by successive application of σ. These p $SGS\,[k,\,1,\,m]$ constitute a (cyclic) $SGS\,[k,\,2,\,m]$ (II 8 g) if no 2-ad occurs in more than one k-ad of the arrangement, that is, here, if no interval occurs in more than one of the ν k-ads $[[i_f]]_g\ (g=1\,\ldots\,.\,\nu)$.

The $SGS\,[p^{n'},\,2,\,p^{nn'}]$ (p a prime, n, n' positive integers) of II 10 f is cyclic.

We proceed to construct a cyclic $SGS\,[k,\,2,\,m]$ for the particular case $k=4$, $p=4\nu+1$ a prime, $m=4\,(3\nu+1)$, $m-1=3p$; $p\neq 3$.

The m indices are $*$ and $m-1=3p$ indices i taken modulo $3v$. Let r_0 be a primitive root of the congruence $x^{p-1}=x^{4\nu}\equiv 1\ (\text{mod. } p)$. Determine $r\ (\text{mod. } 3p)$ by the conditions $r\equiv r_0\ (\text{mod.}\,p)$, $r\not\equiv 0\ (\text{mod.}\,3)$. Determine $s\ (\text{mod.}\,3p)$ by the conditions $s\equiv r^{\nu}\ (\text{mod. } p)$, $s\equiv -1\ (\text{mod. } 3)$. Then $s^2\equiv -1\ (\text{mod. } p)$, $s^2\equiv +1\ (\text{mod. } 3)$, $s^4\equiv +1\ (\text{mod. } 3p)$.

Then when we set

$$i_{fg}\equiv s^{f-1}r^{g-1}\left(\begin{matrix}f=1\,\ldots\,.\,4\\ g=1\,\ldots\,.\,\nu\end{matrix}\right),\ i_{fgv}\equiv i_{fg}+vp\ (v=0,\,1,\,2)\quad (\text{mod. } 3p)$$

we have in the $3\nu+1$ 4-ads

$$[*\ 0\ p\ 2p]\ [[i_f]]_{gv}\qquad\qquad \left(\begin{matrix}g=1\,\ldots\,.\,\nu\\ v=0,\,1,\,2\end{matrix}\right)$$

the first $SGS\,[4,\,1,\,3p+1]$ of a cyclic $SGS\,[4,\,2,\,3p+1]$. In fact, in the first place, the $4\nu=p-1$ indices i_{fg} are (mod. p) respectively congruent to the numbers $1,\,2,\,\ldots\,.\,p-1$ in a certain order, and so the $3\,(p-1)=m-4$ indices i_{fgv} are exactly the m indices lacking $*\ 0\ p\ 3p$. And in the second place, no interval occurs in more than one of the ν 4-ads

$$[[i_f]]_g=[r^{g-1}\ sr^{g-1}\ s^2r^{g-1}\ s^3r^{g-1}].\qquad (g=1\,\ldots\,.\,.\,\nu)$$

The 4-ad $[[i_f]]_g$ has the twelve intervals

$$\pm\left(s-1,\ s(s-1),\ s^2(s-1),\ s^3(s-1)\equiv 1-s^3,\ s^2-1,\ s(s^2-1)\right)r^{g-1}.$$

Of these the only ones divisible by 3 are the four $\pm(1, s)\,r^{g-1}(s^2-1)$. The $4\nu=p-1$ intervals $\pm(1, s)\,r^{g-1}(s^2-1)$ $(g=1\,.\,.\,.\,.\,\nu)$ are exactly the $p-1$ distinct intervals divisible by 3, since $s^2\equiv-1$ (mod. p). The $8\nu=2(p-1)$ intervals $\pm(1, s, s^2, s^3)\,r^{g-1}(s-1)$ $(g=1\,.\,.\,.\,.\,\nu)$ are exactly the $2(p-1)$ distinct intervals divisible neither by 3 nor by p. For the congruence $s^{f-1}r^{g-1}\equiv+s^{f'-1}r^{g'-1}$ (mod. $3p$) or even (mod. p), requires $f=f'$, $g=g'$, while the congruence $s^{f-1}r^{g-1}\equiv-s^{f'-1}r^{g'-1}$ (mod. $3p$) is impossible, since $s^2\equiv-1$ (mod. p), $s^2\equiv+1$ (mod. 3).

For example:

Cyclic $SGS\,[4, 2, 16]$ $r=2$, $s=2$: exhibited in II 10 g.
Cyclic $SGS\,[4, 2, 40]$ $r=2$, $s=8$: exhibited in §7.
Cyclic $SGS\,[4, 2, 52]$ $r=5$, $s=47$: exhibited in §7.

7. *Triple-whist-tournament arrangements* $TWh\,[m]$. At a *table* $|a_i a_j a_k a_l|$ there is no distinction between the two opponents of a player. We introduce the notion: *triple-table* $(a_i a_j a_k a_l)$ at which the pairs of (p) *partners* are $a_i a_j$, $a_k a_l$; (o_1) *opponents of the first kind* are $a_i a_k$, $a_j a_l$; (o_2) *opponents of the second kind* are $a_i a_l$, $a_j a_k$. The four triple-tables

$$(a_i a_j a_k a_l),\ (a_j a_i a_l a_k),\ (a_k a_l a_i a_j),\ (a_l a_k a_j a_i),$$

are identical; a triple-table $(a_i a_j a_k a_l)$ is invariant under the substitutions of the four-group G_4^4, preserving the three pairings (p) $a_i a_j$, $a_k a_l$; (o_1) $a_i a_k$, $a_j a_l$; (o_2) $a_i a_l$, $a_j a_k$. A table $|a_i a_j a_k a_l|$ yields two triple-tables $(a_i a_j a_k a_l)$, $(a_i a_j a_l a_k)$. A 4-ad $[a_i a_j a_k a_l]$ yields six triple-tables

$$\begin{gathered}(a_i a_j a_k a_l),\ (a_i a_k a_l a_j),\ (a_i a_l a_j a_k),\\ (a_i a_j a_l a_k),\ (a_i a_k a_j a_l),\ (a_i a_l a_k a_j);\end{gathered}$$

the pairings (p, o_1, o_2) of one of these triple-tables reappear at the six tables in (all) six different permutations.

I introduce then (as having tactically and also whist-theoretically considerably more interest than the $Wh\,[m]$) the $TWh\,[m]$ defined thus:

$TWh\,[m]$. *A triple-whist-tournament arrangement in* m *players* $TWh\,[m]$ $(m\equiv 0$, mod. 4) *is an arrangement of* $m-1$ *evenings of play at triple-tables in* *such a way that every player meets every other player once as partner, once as opponent of the first kind, and once as opponent of the second kind.*

We have seen that one triple-table gives rise to six triple-tables by permuting in the six possible ways the three players seated say at S, W, E; every such permutation permutes correspondingly the pairings (p, o_1, o_2). If in a given $TWh\,[m]$ we make in every triple-table simultaneously the same permutation of the players seated at S, W, E, we change the $TWh\,[m]$ into another $TWh\,[m]$.

Every $TWh\,[m]$ yields then *six* $TWh\,[m]$. Every $TWh\,[m]$ yields, as it stands, one $Wh\,[m]$, and in all (not six but) *three* $Wh\,[m]$. Thus the term, TRIPLE-*whist-tournament arrangement*, is justified.

What about the existence of $TWh\,[m]$? A $Wh\,[m]$ may or may not yield a $TWh\,[m]$ when we replace its tables by triple-tables, each table by one of its two triple-tables.

A *cyclic* $Wh\,[m]$ yields a *cyclic* $TWh\,[m]$ if we can so replace the tables of the first evening of the cyclic $Wh\,[m]$ by triple-tables that (using the index notation) the interval-systems $(IS)_p$, $(IS)_{o_1}$, $(IS)_{o_2}$ for the pairings (p), (o_1), (o_2) shall each include the $m-2$ possible intervals. Thus, for example, I have written down the cyclic $Wh\,[4]$, $Wh\,[8]$, $Wh\,[16]$ of §2, so that as they stand they yield cyclic $TWh\,[4]$, $TWh\,[8]$, $TWh\,[16]$. On the other hand, neither the cyclic $Wh\,[12]_1$ nor the cyclic $Wh\,[12]_2$ yields a cyclic $TWh\,[12]$. Indeed, in a search intended to be systematic and exhaustive no cyclic $TWh\,[12]$ came to light.

Every $SGS\,[4,\ 2,\ m]$ *gives rise to an associated* $TWh\,[m]$; we let the 4-ad $[a_i\,a_j\,a_k\,a_l]$ yield the first or the second triad of triple-tables written down above.

Every cyclic $SGS\,[4,\ 2,\ m]$ (§6) *gives rise to an associated cyclic* $TWh\,[m]$; the $3\nu+1$ triple-tables of the first evening are

$$(*\ \ 0\ \ p\ \ 2p)\ (i_{1g0}\ i_{2g0}\ i_{3g0}\ i_{4g0})\ (i_{1g1}\ i_{3g1}\ i_{4g1}\ i_{2g1})\ (i_{1g2}\ i_{4g2}\ i_{2g2}\ i_{3g2})\,. \qquad (g=1\,\ldots\,.\ \nu)$$

Thus from the cyclic $SGS\,[4,\ 2,\ m]$ of §6 (II 10 g, h) we have for example:

Cyclic $TWh\,[16]$ associated with $SGS\,[4,\ 2,\ 16]$:

(* 0 5 10) (1 2 4 8) (6 13 9 7) (11 12 14 3).

Cyclic $TWh\,[64]$ associated with $SGS\,[4,\ 2,\ 64]$:

(* 0 21 42)
(1 25 56 58) (2 50 49 53) (3 13 20 57) (6 26 40 51) (12 52 17 39)
(22 14 16 46) (23 7 11 8) (24 41 15 34) (27 61 9 47) (33 38 60 10)
(43 37 4 35) (44 32 29 28) (45 36 55 62) (48 30 5 19) (54 18 31 59).

Cyclic $TWh\,[40]$ associated with $SGS\,[4,\ 2,\ 40]$:

$$\begin{array}{ccc}
 & (*\ 0\ 13\ 26) & \\
(1\ \ 8\ \ 25\ \ 5) & (2\ \ 16\ \ 11\ \ 10) & (4\ \ 32\ \ 22\ \ 20) \\
(14\ \ 38\ \ 18\ \ 21) & (15\ \ 24\ \ 23\ \ 29) & (17\ \ 35\ \ 33\ \ 6) \\
(27\ \ 31\ \ 34\ \ 12) & (28\ \ 36\ \ 3\ \ 37) & (30\ \ 7\ \ 19\ \ 9).
\end{array}$$

Cyclic $TWh\,[52]$ associated with $SGS\,[4,\ 2,\ 52]$:

$$\begin{array}{cccc}
 & (*\ 0\ 17\ 34) & & \\
(1\ \ 47\ \ 16\ \ 38) & (5\ \ 31\ \ 29\ \ 37) & (25\ \ 2\ \ 43\ \ 32) & (23\ \ 10\ \ 11\ \ 7) \\
(18\ \ 33\ \ 4\ \ 13) & (22\ \ 46\ \ 3\ \ 48) & (42\ \ 9\ \ 49\ \ 19) & (40\ \ 28\ \ 24\ \ 27) \\
(35\ \ 21\ \ 30\ \ 50) & (39\ \ 20\ \ 14\ \ 12) & (8\ \ 15\ \ 36\ \ 26) & (6\ \ 41\ \ 44\ \ 45).
\end{array}$$

8. *The* $SGS\,\{4,\ 2,\ m\}$ *associated with a* $TWh\,[m]$. Suppose in a $TWh\,[m]$ we write in place of every triple-table $(a_i\,a_j\,a_k\,a_l)$ the four 4-ids $\{a_i\,a_j\,a_k\,a_l\}$, $\{a_j\,a_i\,a_l\,a_k\}$, $\{a_k\,a_l\,a_i\,a_j\}$, $\{a_l\,a_k\,a_j\,a_i\}$ corresponding to that triple-table and coming out of any one of them by the substitutions of the four-group G_4^4. We obtain from every evening of the $TWh[m]$ obviously a $SGS\{4,\ 1,\ m\}$; these $m-1$ $SGS\,\{4,\ 1,\ m\}$ constitute (II 9 h) a $SGS\,\{4,\ 2,\ m\}$, since two 4-ids never have a 2-id in common. Conversely, a $SGS\,\{4,\ 2,\ m\}$ whose every $SGS\,\{4,\ 1,\ m\}$ contains simultaneously with a 4-id $\{a_i\,a_j\,a_k\,a_l\}$ those four 4-ids determines a $TWh\,[m]$.

Thus the $TWh\,[m]$ occupies a position intermediate between the $SGS\,\{4,\ 2,\ m\}$ and the $SGS\,[4,\ 2,\ m]$, the mediation being effected by the $SGS\,\{4,\ 1,\ 4\}$ dependent upon the G_4^4. Just so in general any $SGS\,\{k,\ 1,\ k\}$ leads to arrangements intermediate between the $SGS\,\{k,\ l,\ m\}$ and the $SGS\,[k,\ l,\ m]$. We have (II 16 e) for the $SGS\,\{k,\ 2,\ m\}$ a composition theorem. When the mediating $SGS\,\{k,\ 1,\ k\}$ depends upon a group G_k^k, this composition effects a composition for the intermediate arrangements.

9. *Given a* $TWh\,[m]^{(a)}$ *and a* $TWh\,[n]^{(b)}$: *to construct a* $TWh\,[mn]^{(c)}$.

We use the m, n, mn letters $a_i\,(i=1\,.\,.\,.\,.\,m)$, $b_j\,(j=1\,.\,.\,.\,.\,n)$, $c_{ij}\begin{pmatrix} i=1\,.\,.\,.\,.\,m \\ j=1\,.\,.\,.\,.\,n \end{pmatrix}$.

We make from the given $TWh\,[m]^{(a)}$ its associated $SGS\,\{4,\ 2,\ m\}^{(a)}$ (§8). We extend this $SGS\,\{4,\ 2,\ m\}^{(a)}$ to a $SGS\,\{\underset{\bullet}{4},\ 2,\ m\}^{(a)}$ by adding as m^{th} $SGS\,\{\underset{\bullet}{4},\ 1,\ m\}^{(a)}$ the m $\underset{\bullet}{4}$-idic sequences $\{a_i\,a_i\,a_i\,a_i\}$ $(i=1\,.\,.\,.\,.\,m.)$

The given $TWh\,[n]^{(b)}$ yields similarly a $SGS\,\{\underset{\cdot}{4},\ 2,\ n\}^{(b)}$.

We compound the $\underset{\cdot}{4}$-ids $\{a_{i_1}a_{i_2}a_{i_3}a_{i_4}\}$, $\{b_{j_1}b_{j_2}b_{j_3}b_{j_4}\}$ into the $\underset{\cdot}{4}$-id $\{c_{i_1j_1}c_{i_2j_2}c_{i_3j_3}c_{i_4j_4}\}$; every $\underset{\cdot}{4}$-id$^{(c)}$ is so obtainable and in only one way.

When we compound in this way every $\underset{\cdot}{4}$-id$^{(a)}$ of a $SGS\,\{\underset{\cdot}{4},\ 1,\ m\}^{(a)}$ with every $\underset{\cdot}{4}$-id$^{(b)}$ of a $SGS\,\{\underset{\cdot}{4},\ 1,\ n\}^{(b)}$, we obtain a $SGS\,\{\underset{\cdot}{4},\ 1,\ mn\}^{(c)}$. Making such composition of the m $SGS\,\{\underset{\cdot}{4},\ 1,\ m\}^{(a)}$ of the $SGS\,\{\underset{\cdot}{4},\ 2,\ m\}^{(a)}$ with the n $SGS\{\underset{\cdot}{4},\ 1,\ n\}^{(b)}$ of the $SGS\,\{\underset{\cdot}{4},\ 2,\ n\}^{(b)}$ we obtain mn $SGS\,\{\underset{\cdot}{4},\ 1,\ mn\}^{(c)}$ constituting a $SGS\,\{\underset{\cdot}{4},\ 2,\ mn\}^{(c)}$ (II 16 b). Discarding the $SGS\,\{\underset{\cdot}{4},\ 1,\ mn\}^{(c)}$ compounded from the m^{th} $SGS\,\{\underset{\cdot}{4},\ 1,\ m\}^{(a)}$ and the n^{th} $SGS\,\{\underset{\cdot}{4},\ 1,\ n\}^{(b)}$ which contains the mn 4-idic sequences $\{c_{ij}c_{ij}c_{ij}c_{ij}\}\begin{pmatrix} i=1\,\ldots\,.\,m\\ j=1\,\ldots\,.\,n\end{pmatrix}$, the only $\underset{\cdot}{4}$-ids not 4-ids of the $SGS\,\{\underset{\cdot}{4},\ 2,\ mn\}^{(c)}$, we obtain (II 16 e) a $SGS\,\{4,\ 2,\ mn\}^{(c)}$. Every $SGS\,\{4,\ 1,\ mn\}^{(c)}$ of this $SGS\,\{4,\ 2,\ mn\}^{(c)}$ contains simultaneously with a 4-id $\{c_{i_1j_1}\,c_{i_2j_2}\,c_{i_3j_3}\,c_{i_4j_4}\}$ its four conjugate 4-ids under the group G_4^4 of position-permutations, and is thus associated (§8) with the $TWh\,[mn]^{(c)}$ desired.

This same method serves to compound a $Wh\,[mn]^{(c)}$ from a $TWh\,[m]^{(a)}$ and a $Wh\,[n]^{(b)}$. I proceed to explain more exactly.

10. *The three interlaced* $SGS\,[2,\ 2,\ m]$ *of a* $TWh\,[m]$. A $Wh\,[m]$ by its partnership-pairings as distributed over the $m-1$ evenings exhibits a $SGS\,[2,\ 2,\ m]_p$. Similarly by its three kinds of pairings a $TWh\,[m]$ exhibits three interlaced $SGS\,[2,\ 2,\ m]$, say $SGS[2,\ 2,\ m]_p$, $SGS[2,\ 2,\ m]_{o_1}$, $SGS[2,\ 2,\ m]_{o_2}$. The interlacing is in this fashion: The $m-1$ sets of the three $SGS\,[2,\ 2,\ m]$ are in 1—1—1 correspondence. No two of any three corresponding sets have a 2-ad in common. Any three corresponding sets are so related that if a letter a_i enters them in the 2-ads $[a_i\,a_j]$, $[a_i\,a_k]$, $[a_i\,a_l]$ respectively, they contain respectively the 2-ads $[a_k a_l]$, $[a_l a_j]$, $[a_j a_k]$. Conversely, three (ordered) $SGS\,[2,\ 2,\ m]$ so interlaced determine a $TWh\,[m]$.

In the process (§9) of compounding a $TWh\,[m]^{(a)}$ and a $TWh\,[n]^{(b)}$ into a $TWh\,[mn]^{(c)}$, we obviously compound the three interlaced $SGS\,[2,\ 2,\ m]^{(a)}$ with the three interlaced $SGS\,[2,\ 2,\ n]^{(b)}$ (each system of the first three with its corresponding system of the second three) to form the three interlaced $SGS\,[2,\ 2,\ mn]^{(c)}$. Each pair separately compounds by the process of II 16 e.

11. *Given a* $TWh\,[m]^{(a)}$ *and a* $Wh\,[n]^{(b)}$: *to construct a* $Wh\,[mn]^{(c)}$.

Replace every table $|b_{j_1}\,b_{j_2}\,b_{j_3}\,b_{j_4}|$ of the $Wh\,[n]^{(b)}$ as it stands by the triple-table $(b_{j_1}\, b_{j_2}\, b_{j_3}\, b_{j_4})$. Thus we obtain say a *pseudo-TWh* $[n]^{(b)}$. (If we obtain a *TWh* $[n]^{(b)}$, §8 applies at once.) This pseudo-*TWh* $[n]^{(b)}$ determines and is determined by three interlaced arrangements *SGS* $[2, 2, n]_p^{(b)}$; pseudo-*SGS* $[2, 2, n]_{o_1}^{(b)}$, pseudo-*SGS* $[2, 2, n]_{o_2}^{(b)}$.

We apply a process quite analogous to that of §9 to the *TWh* $[m]^{(a)}$ and the pseudo-*TWh* $[n]^{(b)}$ and obtain a pseudo-*TWh* $[mn]^{(c)}$ yielding when we replace every triple-table by the corresponding table the *Wh* $[mn]^{(c)}$ desired. In fact, the *SGS* $[2, 2, m]_p^{(a)}$ and the *SGS* $[2, 2, n]_p^{(b)}$ compound into a *SGS* $[2, 2, mn]_p^{(c)}$. The *SGS* $[2, 2, m]_{o_1}^{(a)}$ and the pseudo-*SGS* $[2, 2, n]_{o_1}^{(b)}$ compound into a pseudo-*SGS* $[2, 2, mn]_{o_1}^{(c)}$, and similarly the two arrangements o_2 yield a pseudo-*SGS* $[2, 2, mn]_{o_2}^{(c)}$. We must prove that every 2-ad $[c_{i_1j_1}\, c_{i_2j_2}]$ $(i_1j_1 \neq i_2j_2)$ occurs twice in the two pseudo-*SGS* $[2, 2, mn]_{o_1, o_2}^{(c)}$. Now every 2-ad $[a_{i_1}\, a_{i_2}]$ $(i_1 \neq i_2)$ occurs once in *SGS* $[2, 2, m]_{o_1}^{(a)}$ and once in *SGS* $[2, 2, m]_{o_2}^{(a)}$. And every 2-ad $[b_{j_1}\, b_{j_2}]$ $(j_1 \neq j_2)$ occurs twice in the two pseudo-*SGS* $[2, 2, n]_{o_1, o_2}^{(b)}$. These two remarks show, the first, that every 2-ad $[c_{i_1j}\, c_{i_2j}]$ $(i_1 \neq i_2)$ occurs once in the pseudo-*SGS* $[2, 2, mn]_{o_1}^{(c)}$ and once in the pseudo-*SGS* $[2, 2, mn]_{o_2}^{(c)}$; the second, that every 2-ad $[c_{ij_1}\, c_{ij_2}]$ $(j_1 \neq i_2)$ occurs twice in the two pseudo-*SGS* $[2, 2, mn]_{o_1, o_2}^{(c)}$; and the two together, that every 2-ad $[c_{i_1j_1}\, c_{i_2j_2}]$ $(i_1 \neq i_2, j_1 \neq j_2)$ occurs twice in the two pseudo-*SGS* $[2, 2, mn]_{o_1, o_2}^{(c)}$.

The composition process (§§9, 11) is *distributive* as to the triple-tables or tables of the *TWh* $[m]^{(a)}$ and of the *TWh* $[n]^{(b)}$ or *Wh* $[n]^{(b)}$. Hence if in the *TWh* $[m]^{(a)}$ certain m' players$^{(a)}$ a_{i_k} $(k = 1 \ldots . m')$ engage in a *TWh* $[m']^{(a)}$ (if at all in a *Wh* $[m']^{(a)}$ it will be in a *TWh* $[m']^{(a)}$)—whose evenings need not be *as wholes* parts of certain evenings of the *TWh* $[m]^{(a)}$—and if in the *TWh* $[n]^{(b)}$ or *Wh* $[n]^{(b)}$ certain n' players$^{(b)}$ b_{j_l} $(l = 1 \ldots . n')$ engage in a *TWh* $[n']^{(b)}$ or *Wh* $[n']^{(b)}$, then in the compound *TWh* $[mn]^{(c)}$ or *Wh* $[mn]^{(c)}$ the corresponding $m'n'$ players$^{(c)}$ $c_{i_kj_l}\left(\begin{matrix} k = 1 \ldots . m' \\ l = 1 \ldots . n' \end{matrix}\right)$ will engage in a *TWh* $[m'n']^{(c)}$ or *Wh* $[m'n']^{(c)}$. For the purposes of the concluding remark just made, it is proper to consider each player a_i or b_j as engaging in a *TWh* [1] by himself; that is, the players c_{ij_l} $(l = 1 \ldots . n'$; i any) engage in a *TWh* $[n']^{(c)}$ or *Wh* $[n']^{(c)}$, and the players c_{i_kj} $(k = 1 \ldots . m'$; j any) in a *TWh* $[m']^{(c)}$.

If the arrangements to be compounded are associated with a *SGS* $[4, 2, m]^{(a)}$, *SGS* $[4, 2, n]^{(b)}$ respectively, the resulting arrangement will be associated with a *SGS* $[4, 2, mn]^{(c)}$.

12. *Direct construction of a system of associated* $SGS\,\{4, 2, 4^n\}$, $TWh\,[4^n]$, $SGS\,[4, 2, 4^n]$ (*n any positive integer*).

We use the $SGS\,\{\underset{\cdot}{4}, 2, 4\}$ given in II 15 h, viz. the four letters being 1, 2, 3, 4,

$$\begin{array}{cccc}
\{1\,1\,1\,1\} & \{2\,2\,2\,2\} & \{3\,3\,3\,3\} & \{4\,4\,4\,4\}, \\
\{1\,2\,3\,4\} & \{2\,1\,4\,3\} & \{3\,4\,1\,2\} & \{4\,3\,2\,1\}, \\
\{1\,3\,4\,2\} & \{2\,4\,3\,1\} & \{3\,1\,2\,4\} & \{4\,2\,1\,3\}, \\
\{1\,4\,2\,3\} & \{2\,3\,1\,4\} & \{3\,2\,4\,1\} & \{4\,1\,3\,2\}.
\end{array}$$

This $SGS\,\{\underset{\cdot}{4}, 2, 4\}$ has its sets s_1 invariant under the four-group G_4^4, and is itself invariant under the alternating group G_{12}^4; the G_4^4 is self-conjugate under the G_{12}^4.

We work with the 4^n letters $a_{f_1 \ldots. f_j \ldots. f_n}$, where the n indices $f_j\,(j=1 \ldots . n)$ have independently the values 1, 2, 3, 4. We denote by F the n-id of indices $F = \{f_1 \ldots . f_j \ldots . f_n\}$.

We construct first a $S\,\{\underset{\cdot}{4}, 2, 4^n\}$. This system contains those $\underset{\cdot}{4}$-ids $\{a_{F_1} a_{F_2} a_{F_3} a_{F_4}\}$, $F_i = \{f_{i1} \ldots . f_{ij} \ldots . f_{in}\}$ $(i = 1, 2, 3, 4)$ for which the $\underset{\cdot}{4}$-id $F^j = \{f_{1j} f_{2j} f_{3j} f_{4j}\}$ is for every j $(j = 1 \ldots . n)$ a 4-id of the auxiliary $SGS\,\{\underset{\cdot}{4}, 2, 4\}$.

The $\underset{\cdot}{4}$-id $\{a_{F_1} a_{F_2} a_{F_3} a_{F_4}\}$ has four different letters and is a 4-id, or else it is a 4-idic sequence $\{a_F a_F a_F a_F\}$. Discarding these sequences, we have a $S\,\{4, 2, 4^n\}$.

We exhibit this $S\,\{4, 2, 4^n\}$ as a $SGS\,\{4, 2, 4^n\}$. From the 4-id $\{a_{F_1} a_{F_2} a_{F_3} a_{F_4}\}$ by its 4-ids F^j $(j=1 \ldots n)$ and any n substitutions σ_j $(j=1 \ldots n)$ of the G_4^4 we obtain by applying σ_j to F^j $(j = 1 \ldots . n)$ another 4-id of the $S\,\{4, 2, 4^n\}$. In this way one 4-id determines in all 4^n 4-ids constituting a $S\,\{4, 1, 4^n\}$; this $S\,\{4, 1, 4^n\}$ is in like manner determined by any one of the 4^n 4-ids.

Since the auxiliary $SGS\,\{\underset{\cdot}{4}, 2, 4\}$ is invariant when to every $\underset{\cdot}{4}$-id is applied any the same *position*-permutation π of the G_{12}^4, the $SGS\,\{4, 2, 4^n\}$ is likewise invariant. A 4-id $\{a_{F_1} a_{F_2} a_{F_3} a_{F_4}\}$ gives rise thus in all to twelve 4-ids $\{a_{F_1} a_{F_2} a_{F_3} a_{F_4}\}$ containing the same four letters. The four 4-ids $\{\ \}_\pi$, where π belongs to the G_4^4, lie in the $SGS\,\{4, 2, 4^n\}$ determined by $\{a_{F_1} a_{F_2} a_{F_3} a_{F_4}\}$. The other eight lie by fours in the two $SGS\,\{4, 2, 4^n\}$ determined by $\{a_{F_1} a_{F_3} a_{F_4} a_{F_2}\}$, $\{a_{F_1} a_{F_4} a_{F_2} a_{F_3}\}$ respectively.

Thus the $4^n - 1$ $SGS\,\{4, 1, 4^n\}$ of the $SGS\,\{4, 2, 4^n\}$ separate into $\frac{1}{3}(4^n - 1)$ triads of $SGS\,\{4, 1, 4^n\}$.

Further replacing every 4-id $\{a_{F_1} a_{F_2} a_{F_3} a_{F_4}\}$ by the corresponding triple-table $(a_{F_1} a_{F_2} a_{F_3} a_{F_4})$ and discarding repetitions we have a $TWh\,[4^n]$. Again, replacing every 4-id by the corresponding 4-ad and discarding repetitions we have a $SGS\,[4,\ 2,\ 4^n]$. The $TWh\,[4^n]$ is associated with the $SGS\,[4,\ 2,\ 4^n]$.

The $TWh\,[4^n]$ whose direct construction has just been explained is exactly the $TWh\,[4^n]$ which would have been obtained from the unique $TWh\,[4]$ by successive composition (§8) with itself, obtaining thus $TWh\,[4]$, $TWh\,[4^2]$, ... $TWh\,[4^n]$. The $TWh\,[4^n]$ contains many $TWh\,[4^{n_1}]$ for $n_1 = 1,\ 2, \ldots.\ n-1$; for instance, the 4^{n_1} players $a_F = a_{f_1 \ldots\ldots f_n}$ whose complexes F of indices have any certain $n - n_1$ indices in common, obviously engage in a $TWh\,[4^{n_1}]$. The question, however, of *all* $TWh\,[4^{n_1}]$ included in this $TWh\,[4^n]$ depends upon a similar question for the $SGS\,\{4,\ 2,\ 4^n\}$, which will not now be taken up.

13. *Summarized results of this study of Wh* $[m]$ *and TWh* $[m]$.

The new notions—triple-table and triple-whist-tournament arrangement $TWh\,[m]$—and the compositions thereon dependent—(§9) of a $TWh\,[m_1 m_2]$ from given $TWh\,[m_1]$, $TWh\,[m_2]$; (§11) of a $Wh\,[m_1 m_2]$ from given $TWh\,[m_1]$, $Wh\,[m_2]$—are of principal importance.

Let us denote for brevity by

n any positive integer,
d any integer either 0 or $\geqq 2$,
t the integer $t = \frac{1}{4}(3p + 1)$ where p is any prime of the form $p = 4\nu + 1$,
t_n the product of any n integers t,

Then we have obtained directly: (a) $TWh\,[4^n]$ (§12); (b) $TWh\,[8]$ (§7); (c) $TWh\,[4t]$ (§7), whence by composition (§9), (d) $TWh\,[2^{n+1}]$; (e) $TWh\,[4^n t_n]$; (f) $TWh\,[2^{2n+d} t_n]$.

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